

SPECTRAL ANALYSIS OF POLYNOMIAL POTENTIALS AND ITS RELATION WITH ABJ/M-TYPE THEORIES

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ABSTRACT. We obtain a general class of polynomial potentials for which the Schrödinger operator has a discrete spectrum. This class includes all the scalar potentials in membrane, 5-brane, p-branes, multiple M2 branes, BLG and ABJM theories. We provide a proof of the discreteness of the spectrum of the associated Schrödinger operators. This is a first step in order to analyze BLG and ABJM supersymmetric theories from a non-perturbative point of view.

1. INTRODUCTION

There is an intense activity in the spectral characterization of ABJM-type theories at perturbative level. These theories belong to a class of superconformal Chern-Simons gauge theories in three dimensions with $\mathcal{N} = 6$ supersymmetry [1]. The gauge group is $U(N) \times U(N)$ with Chern-Simons level k . The case with gauge group $U(N) \times U(M)$ with different gauge groups $N \neq M$, also called ABJ was considered in [2]. ABJM theories are special cases of the Gaiotto-Witten theories [3] i.e. Superconformal Chern-Simons Theories with $\mathcal{N} = 4$, in which the supersymmetry is enhanced to $\mathcal{N} = 6$. For the case $N = 2$, the number of supersymmetries is enhanced to $\mathcal{N} = 8$ and it corresponds to the BLG theory, [4],[5] and [6]. In these papers, the fields are evaluated on a 3-algebra with positive inner metric, in terms of a unique finite dimensional gauge group $SO(4)$ with twisted Chern-Simons terms. The ABJM theory or at least a sector of it, can also be recovered from the 3-algebra formulation by relaxing the antisymmetry condition in its structure constants[7].

The interest of these ABJ-like theories is double: on one hand they represent further evidence of the duality AdS_4/CFT_3 [8]. This duality opens an interesting window that allows to compute different aspects of condensed matter in the strong coupling limit in the fields of superconductivity, semiconductors, and so on, unreachable today by other means. For recent reviews on this interesting topic, see [9],[10]. On the other hand these calculations also provide results about integrability and finiteness properties of these superconformal Chern-Simons theories in the strong coupling regime. There have been results in this regard recently, see for example [11, 12, 13, 14, 15, 16]. For all these reasons, any non perturbative results related to these theories are of obvious interest.

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An important aspect of super-membranes, super 5-branes and supersymmetric multiple-M2 branes refers to the quantum stability of the theory and the validity of the Feynman kernel. A natural way to proceed is to formulate the theory on a compact base manifold, perform then a regularization of the theory in terms of an orthonormal basis and analyze properties of the spectrum of the associated Schrödinger operator. This procedure, to start with a field theory and analyze its properties by going to a regularized model, has been very useful in field theory, although relevant symmetries of the theory may be lost in the process. In the case of the $D = 11$ supermembrane [17], an important property of the regularization is that the area preserving diffeomorphism, the residual gauge symmetry of the supermembrane in the light cone gauge, gives rise to a $SU(N)$ gauge symmetry of the regularized model [18, 19]. The gauge symmetry of the field theory is then 'represented' as the $SU(N)$ gauge symmetry of the regularized model and it is not lost in the reduction to finite degrees of freedom. The quantum properties of the regularized model is then determined from the Schrödinger operator $-\Delta + V(x)$ + fermionic terms, where the bosonic potential $V(x)$ has the expression

$$(1) \quad V(x) = \sum_i \left[P_i(x) \right]^2.$$

$P_i(x)$ is a homogeneous polynomial on the configuration variables $x \in \mathbb{R}^n$. In the membrane theory $P_i(x)$ are of degree two.

An important aspect of $V(x)$, which determines the spectrum of the associated Schrödinger operator, is the algebraic variety of zero potential which extends to infinity on configuration spaces and the behavior of the potential along that variety. In the case of the (bosonic) membrane the distance between the walls of the valleys along the zero variety goes to zero as we approach infinity, and this was interpreted in [20] as the main reason explaining the discreteness of the spectrum of the membrane hamiltonian: the wavefunction cannot escape to infinity. The potential in the transverse directions to the valleys behaves as the potential of an harmonic oscillator. The proof of the discreteness was done in [21] where a bound

$$(2) \quad \langle \Psi, H\Psi \rangle \geq \langle \Psi, \lambda\Psi \rangle,$$

in terms of a function $\lambda(x)$, with $\lambda(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ was obtained. A proof of the discreteness of the spectrum of the membrane, following an extension of the Barry Simon argument for a toy model $V(x) = x^2 y^2$ in two dimensions, was presented in [22].

An important remark to be mentioned is that supermembrane theory is an example of a field theory over a compact manifold which (at the regularized level) has continuous spectrum [20]. There are several related toy models which also have continuous spectrum, see for example [20]. It is only when the supermembrane is restricted by certain topological conditions, non-trivial central charges, that the spectrum becomes discrete, with eigenvalues accumulating at infinity [23, 24, 25, 26, 27]. In order to analyse with more precision the supermembrane and super 5-brane potentials, and even more complicated potentials as in the BLG and ABJM theories it is very useful to consider a necessary and sufficient condition to have a discrete spectrum. This was achieved by A. M. Molchanov [28] and more recently extended by V. Maz'ya and M. Schubert [29]. It makes use of the mean value of the potential, in the sense of Molchanov, on a star shaped cell \mathcal{G}_d , of diameter d . The spectrum is discrete if and only if the mean value of the potential goes to infinity when the

distance from \mathcal{G}_d to a fixed point on configuration space goes to infinity in all possible ways. The potential is assumed to be locally integrable and bounded from below.

The mean value in the sense of Molchanov, for the membrane theory, was obtained in [22] in terms of a strictly positive definite inertia tensor for the membrane. As a consequence the spectrum of the hamiltonian of the membrane theory (a regularized $SU(N)$ model) is discrete. Estimates of the eigenvalues may also be obtained by looking at the mean value at finite distances. These estimations are also useful to characterize the mass gap of Yang-Mills theories in the slow mode regime. As an example, in [22] it was obtained a bound for the $3 + 1D$ $SU(3)$ hamiltonian of Yang-Mills theories in the slow mode regime in terms of a hamiltonian whose spectrum and eigenvalues are known, and its eigenfunctions are expressed in terms of Bessel functions .

An analysis of the spectrum of the $D = 11$, 5-brane in the light cone gauge, using the Molchanov, Maz'ya and Schubin theorem was performed in [30],[31]. The spectrum of the hamiltonian is also discrete. However, these results do not apply directly to the BLG and ABJM multi M2 brane theories. A common property of the potential in all cases is the form (1), where $P_i(x)$ have different expressions for each theory. The potential is always a homogeneous polynomial on the configuration variables. The first point to notice is that the form (1) of the potential does not imply discreteness of the spectrum of the hamiltonian. One example, in \mathbb{R}^3 is the following

$$\begin{aligned} (3) \quad h &= -\Delta + V_3 \\ (4) \quad V_3 &= |x - y|^\alpha + |y - z|^\beta + |x - z|^\gamma, \end{aligned}$$

where α, β, γ are any real positive numbers. h has a continuous spectrum. More precisely, the essential spectrum is non-trivial. We say that the spectrum is discrete when the bottom of the essential spectrum is at infinity. All quasi-eigenvectors are then eigenvectors.

In this paper, we obtain a general class of polynomials $P_i(x)$ in (1) for which the Schrödinger operator $-\Delta + V(x)$ has a discrete spectrum and provide a proof of it. This class includes all potentials in membrane, 5-brane, p-branes, multiple M2-branes, BLG and ABJM theories. This is a first step in order to analyse BLG and ABJM super-symmetric theories from a non-perturbative point of view. In section 2, we present preliminary results in particular we describe the Molchanov approach to the analysis of the spectrum. In sections 3 and 4 we present the new results. In section 5 we discuss the application to the BLG and ABJM theories. Finally, in section 6 we present our conclusions.

2. PRELIMINARY RESULTS

K. Friedrichs (see [29] for further references) proved that the spectrum of the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^n)$ with a locally integrable potential V is discrete provided $V(X) \rightarrow \infty$ as $|X| \rightarrow \infty$. This is a sufficiency condition for the discreteness of the spectrum of the Schrödinger operator but, of course it is not necessary. In order to understand the quantum properties of the membrane, supermembrane, 5-brane and multiple brane theories and as a consequence of Yang-Mills theory it is useful to look for a necessary and sufficient condition that ensures

the discreteness of the spectrum. That condition, in terms only of properties of the potential was discovered by A. M. Molchanov [28] and more recently extended by Maz'ya and Schubin [29] and makes use of the mean value, in the sense of Molchanov, of the potential on a star shaped set when the distance from the set to an origin goes to infinity. It is naturally related to the Friedrichs condition, but by taking a mean value of the potential on a cell one obtains also a necessary condition. We will only state the V. Maz'ya and M. Schubin generalization of Molchanov theorem [29]. Let us give first some definitions involved in the formulation of the theorem.

Definition 1. Let $n \geq 2$, $F \subset \mathbb{R}^n$ be compact, and $Lip_c(\mathbb{R}^n)$ the set of all real-valued functions with compact support satisfying a uniform Lipschitz condition in \mathbb{R}^n . Then the Wiener's capacity of F is defined by

$$\text{cap}(F) = \text{cap}_{\mathbb{R}^n}(F) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \mid u \in Lip_c(\mathbb{R}^n), u|_F = 1 \right\}$$

In physical terms the capacity of the set $F \subset \mathbb{R}^n$ is defined as the electrostatic energy over \mathbb{R}^n when the electrostatic potential is set to 1 on F .

Definition 2. Let $\mathcal{G}_d \subset \mathbb{R}^n$ be an open, bounded and star-shaped set of diameter d , let $\gamma \in (0, 1)$. The *negligibility class* $\mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$ consists of all compact sets $F \subset \overline{\mathcal{G}_d}$ satisfying $\text{cap}(F) \leq \gamma \text{cap}(\overline{\mathcal{G}_d})$.

Balls and cubes in \mathbb{R}^n are useful examples of such \mathcal{G}_d . In what follows we denote the ball of diameter d and center x by $B_d(x)$ and the n -dimensional Lebesgue measure by $\text{Vol}(\cdot)$.

Theorem 1 (Maz'ya and Shubin). *Let $V \in L^1_{loc}(\mathbb{R}^n)$, $V \geq 0$.*

Necessity: If the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^n)$ is discrete then for every function $\gamma : (0, +\infty) \rightarrow (0, 1)$ and every $d > 0$

$$(5) \quad \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \int_{\mathcal{G}_d \setminus F} V(x) dx \rightarrow +\infty \quad \text{as } \mathcal{G}_d \rightarrow \infty.$$

Sufficiency: Let a function $d \mapsto \gamma(d) \in (0, 1)$ be defined for $d > 0$ in a neighborhood of 0 and satisfying

$$\limsup_{d \downarrow 0} d^{-2} \gamma(d) = +\infty.$$

Assume that there exists $d_0 > 0$ such that (5) holds for every $d \in (0, d_0)$. Then the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^n)$ is discrete.

Remark 2. *It follows from the previous theorem that a necessary condition for the discreteness of spectrum of $-\Delta + V$ is*

$$(6) \quad \int_{\mathcal{G}_d} V(x) dx \rightarrow \infty \quad \text{as } \mathcal{G}_d \rightarrow \infty.$$

The following lemma is very useful tool in the next sections [22].

Lemma 3. *For each given $\mathcal{G}_d = \mathcal{G}_d(x_0)$,*

$$c_d := \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \text{Vol}(\mathcal{G}_d \setminus F) > 0.$$

Proof. Let $V(x) = |x|$. Then by Friedrichs theorem the spectrum of $-\Delta + V$ is discrete, so by theorem 1 we have

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \int_{\mathcal{G}_d \setminus F} V(x) dx \rightarrow \infty \quad \text{as } |x_0| \rightarrow \infty.$$

Now $\int_{\mathcal{G}_d \setminus F} V(x) dx \leq (|x_0| + d) \text{Vol}(\mathcal{G}_d \setminus F)$ implies that

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \int_{\mathcal{G}_d \setminus F} V(x) dx \leq (|x_0| + d) \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \text{Vol}(\mathcal{G}_d \setminus F),$$

from which follows that $\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \text{Vol}(\mathcal{G}_d \setminus F) > 0$, as we claimed. \square

The following proposition extends the result of B. Simon in [32] which is used also as a toy model for the membrane in [20].

Proposition 4. *Let $V(x) = \prod_{k=1}^n |x_k|^{\alpha_k}$, where $\alpha_k > 0$ for all $k = 1, 2, \dots, n$. Then the spectrum of the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^n)$ is discrete.*

See[22] for a proof using Molchanov ideas.

3. UNIFORMLY BOUNDED BASIS

In this section we prove that the orthonormal basis for polynomials with respect to the inner product over Ω_F is uniformly bounded, independently of F in \mathcal{G}_D . This result will be used in the main proof of the paper.

Proposition 5. *Let $\mathbb{R}[x_1, \dots, x_M]$ be the ring of polynomials over \mathbb{R} in M indeterminate and let $\mathcal{P} = \text{span}\{P_k(x) \in \mathbb{R}[x_1, \dots, x_M], k = 1, \dots, N\}$ be a subspace of dimension N . Let $F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$ and $\Omega_F = \mathcal{G}_d \setminus F$, then*

$$\|P_k\|_{\Omega_F}^2 := \int_{\Omega_F} P_k^2(x) dx > 0, \quad \text{for all } k.$$

Proof. There exists $c_d > 0$ such that $\text{Vol}(\mathcal{G}_d \setminus F) \geq c_d$, hence for each $F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$, there exists a non empty open ball $B_F \subset \Omega_F$

Suppose that $\|P_k\|_{\Omega_F}^2 = 0$. Then $P_k|_B(x) \equiv 0$ (here $P_k|_B$ stands for the restriction of P_k to the set B) for any open ball $B \subset \Omega_F$. In particular, $P_k|_{B_F}(x) \equiv 0$.

As P_k is not the zero polynomial, we have $P_k(x) = a_\alpha x^\alpha + (\text{lower order terms})$ with $a_\alpha \neq 0$ and $|\alpha| \geq 0$. Thus, $\frac{\partial^{|\alpha|}}{\partial x^\alpha} P_k|_{B_F}(x) = a_\alpha \alpha! \neq 0$, which contradicts the fact that $P_k|_{B_F}(x) \equiv 0$. Therefore $\|P_k\|_{\Omega_F}^2 > 0$. \square

Remark 6. *The argument in the proof is essentially that on a bounded set, the Lebesgue measure of the zero set of a non zero polynomial is zero.*

Let $\mathcal{B} = \{P_k(x)\}_{k=1}^N$ be a basis of \mathcal{P} (we are keeping the previous notations). Following the Gram-Schmidt process, with respect to the inner product

$$(f, g)_F := \int_{\Omega_F} f(x)g(x)dx \quad (f, g \in C(\mathcal{G}_d)),$$

we can get an orthonormal basis $\{\varphi_m^F(x)\}_{m=1}^N$ for the space \mathcal{P} . It is clear that we can write $\varphi_m^F(x) = \sum_{k=1}^m b_{mk}^F P_k(x)$, i.e., $b_{mk}^F = 0$ if $k > m$.

Now we can state the main result of this section:

Theorem 7. *There exists $C > 0$ independent of F such that $|\varphi_m^F(x)| \leq C$ for all $x \in \Omega_F$ and all $F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$.*

In order to prove this theorem we need to establish the following preliminaries results.

In the previous notation, let $b_m^F = (b_{m1}^F, \dots, b_{mm}^F, 0, \dots, 0)^\top \in \mathbb{R}^N$. Then

(1) $\{b_m^F\}_{k=1}^N$ is a basis of \mathbb{R}^N and the application $P_m^F \mapsto b_m^F$ defines an isomorphism between \mathcal{P} and \mathbb{R}^N .

(2) Moreover, if $\Phi^F \in \mathbb{R}^{N \times N}$ is the matrix given by $\Phi_{kj}^F = \int_{\Omega_F} P_k(x) P_j(x) dx$,

then

(2.1) $\delta_{mn} = (\varphi_m^F, \varphi_n^F)_F = \langle \Phi^F b_m^F, b_n^F \rangle := (\Phi^F b_m^F)^\top b_n^F$ (the Kronecker delta).

(2.2) Φ^F is symmetric and positive definite. Hence the eigenvalues of Φ^F are positive.

Proof. These assertions follow from straightforward arguments. Let $u \in \mathbb{R}^N$ then $u = \sum_{k=1}^N a_k b_k^F$. The last statement of (2.2) follows from

$$\langle \Phi^F u, u \rangle = \sum_{m,n=1}^N a_m a_n \langle \Phi^F b_m^F, b_n^F \rangle = \sum_{m=1}^N a_m^2 \geq 0,$$

with equality only in the trivial case. \square

Lemma 8. *Let $\sigma(\Phi^F)$ be the spectrum of Φ^F , and let $\lambda_F := \min \sigma(\Phi^F)$. Then*

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \lambda_F > 0.$$

Proof. Suppose that $\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \lambda_F = 0$. Then there is a sequence of matrices $\Phi^{F_1}, \Phi^{F_2}, \dots$ with $F_n \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$ and there are sequences of eigenvalues and eigenvectors $\lambda_{F_1}, \lambda_{F_2}, \dots$ and u_1, u_2, \dots , respectively, with $\Phi^{F_n} u_n = \lambda_{F_n} u_n$, and such that

$$\lambda_{F_n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

I. Without loss of generality, we can suppose that $\|u_n\| = 1$ for all n . Then, using that the unit sphere is compact in finite dimensions, there exists a convergent subsequence with a limit unit vector u_0 . In order to simplify notation, suppose that $u_n \rightarrow u_0$. Let $u_n = (u_{n1}, \dots, u_{nN})^\top$, $n = 0, 1, 2, \dots$, and let $\psi_n = \sum_{k=1}^N u_{nk} P_k$. Then

$$\|\psi_n\|_{\Omega_{F_n}}^2 = \int_{\Omega_{F_n}} \left(\sum_{k=1}^N u_{nk} P_k \right)^2 dx = \langle \Phi^{F_n} u_n, u_n \rangle = \lambda_{F_n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

II. Let $\vec{P}(x) = (P_1(x), \dots, P_N(x))^\top$, and let $\psi(u, x) := \langle u, \vec{P}(x) \rangle = u \cdot \vec{P}(x) = \sum_{k=1}^N w_k P_k(x)$ (where $u = (w_1, w_2, \dots, w_N)$). Then $\psi(u, x)$ is a polynomial and $\psi(u, x) \in \mathcal{P}$. Therefore the zero set of $\psi(u, x)$, namely $Z(\psi(u, x)) = \{x \in \mathbb{R}^N : \psi(u, x) = 0\}$ has measure zero for all $u \in \mathbb{R}^N$. Let \mathbf{N}_0 be an open neighborhood of $Z(\psi(u_0, x)) \cap \mathcal{G}_d$ with measure $c_d/2$ (or more).

$\psi(u_0, x)^2$ is a continuous function, hence on the compact set $\overline{\mathcal{G}_d \setminus \mathbf{N}_0}$ it has a minimum $m^2 \neq 0$. We denote $M^2 = \int_{\mathcal{G}_d} \|\vec{P}(x)\|^2 > 0$. We have, recalling that

$$\psi_n = \psi(u_n, x),$$

$$\begin{aligned} \|\psi_n\|^2_{\Omega_{F_n}} &\geq \|\psi_n\|^2_{\Omega_{F_n} \setminus \mathbf{N}_0} = \|\psi(u_0, x) + \psi(u_n - u_0, x)\|^2_{\Omega_{F_n} \setminus \mathbf{N}_0} \geq \\ &\quad \left| \|\psi(u_0, x)\|_{\Omega_{F_n} \setminus \mathbf{N}_0} - \|\psi(u_n - u_0, x)\|_{\Omega_{F_n} \setminus \mathbf{N}_0} \right|^2. \end{aligned}$$

and

$$\|\psi(u_n - u_0, x)\|^2_{\Omega_{F_n} \setminus \mathbf{N}_0} \leq \|u_n - u_0\|^2 M^2.$$

Consequently,

$$\|\psi_n\|^2_{\Omega_{F_n} \setminus \mathbf{N}_0} \geq \frac{1}{8} m^2 c_d$$

for all u_n such that $\|u_n - u_0\|^2 \leq \frac{1}{8} \frac{m^2}{M^2} c_d$, which is a contradiction with conclusion of (I), which was a consequence of the assumption $\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_0, R^M)} \lambda_F = 0$.

Therefore

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \lambda_F > 0.$$

□

Corollary 9. *There exists $K > 0$ independent of $F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$ such that $\|b_m^F\| \leq K$ for all $m = 1, \dots, N$.*

Proof. Let $K_1 = \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \lambda_F$. As Φ^F is symmetric, there exists $S_F \in \mathbb{R}^{N \times N}$ orthogonal such that $\Phi^F = S_F^\top D_F S_F$ where $D_F = \text{diag}(\lambda_1, \dots, \lambda_N)$ with $\{\lambda_k\}_{k=1}^N = \sigma(\Phi^F)$.

Fix m and let $w = S_F b_m^F = (w_j)^\top$. Then $\|w\| = \|b_m^F\|$ and

$$1 = (\varphi_m^F, \varphi_m^F)_F = \langle \Phi^F b_m^F, b_m^F \rangle = (S_F b_m^F)^\top D_F S_F b_m^F = \sum_{j=1}^N \lambda_j w_j^2 \geq \lambda_F \|b_m^F\|^2 \geq K_1 \|b_m^F\|^2.$$

$$\text{Hence } \|b_m^F\| \leq \frac{1}{\sqrt{K_1}} := K.$$

□

Now we can prove the main theorem.

Proof. (Theorem 7) There exists $C_0 > 0$ such that $|P_k(x)| \leq C_0$ for all $k = 1, \dots, N$ and all $x \in \mathcal{G}_d$. Let $x \in \Omega_F$ and let K be the constant in the previous corollary. Then

$$|\varphi_m^F(x)| = \left| \sum_{k=1}^m b_{mk}^F P_k(x) \right| \leq \sum_{k=1}^m |b_{mk}^F| |P_k(x)| \leq m \|b_m^F\| C_0 \leq N K C_0 := C.$$

□

4. DISCRETENESS OF THE SPECTRUM OF SOME SCHRÖDINGER OPERATORS WITH POLYNOMIAL POTENTIALS.

In this section we prove two propositions ensuring the discreteness of the spectrum of some Schrödinger operators with positive polynomial potentials.

Proposition 10. *Let*

$$(7) \quad V(X) = \sum_{j=1}^J P_j^2(X)$$

Where P_j belong to the ring of polynomials over R in M variables and span a nontrivial subspace of it. If the Schrodinger operator $-\Delta + V(X)$ has discrete spectrum in $L^2(R^M)$, then the operator $-\Delta + \sqrt{V(X)}$ has also discrete spectrum in $L^2(R^M)$.

Proof. Let $\mathcal{G}_d = \mathcal{G}_d(X_0) \subset \mathbb{R}^M$ be a ball centered at X_0 and radius $d > 0$, let $F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$. We decompose $X = X_0 + \xi$ for all X in the cell $\mathcal{G}_d \setminus F$. Let Ω_F be the set of all such ξ . Then the necessary condition of Theorem 1 implies that

$$(8) \quad \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \int_{\Omega_F} V(X_0, \xi) d\xi \longrightarrow \infty \text{ as } |X_0| \longrightarrow \infty$$

We can rewrite the potential as

$$(9) \quad V(X) = \sum_{j=1}^J P_j^2(X_0, \xi)$$

where $P_j(X_0, \xi)$, are polynomials in ξ with coefficients depending on X_0 . Let us denote by N the dimension of the subspace span by $P_j(X_0, \xi)$ with $j = 1, \dots, J$. From this set we consider N independent polynomials, and following the Gram-Schmidt process, whit respect to the inner product

$$(10) \quad (f, g)_F = \int_{\Omega_F} f(\xi)g(\xi) d\xi$$

we can get an orthonormal basis $\varphi_k^F(\xi)$ with $k = 1, \dots, N$ for the subspace span by $P_j(X_0, \xi)$, $j = 1, \dots, J$. It is possible to write

$$P_j(X_0, \xi) = \sum_{k=1}^N a_{jk}(X_0) \varphi_k^F(\xi) \quad j = 1, \dots, J.$$

where $a_{jk}(X_0)$ depends on the set Ω_F .

Let us denote by $M_F > 0$ a uniform bound for $\varphi_k^F(\xi)$ such that $|\varphi_k^F(\xi)| \leq M_F$ for all k and all $\xi \in \Omega_F$.

We have,

$$(11) \quad \|P_j\|_{\Omega_F}^2 := \int_{\Omega_F} P_j^2(X_0, \xi) = \sum_{k=1}^N a_{jk}^2(X_0) \text{ and } \int_{\Omega_F} V d\xi = \sum_j \|P_j\|_{\Omega_F}^2$$

Then using $\left(\sum_{k=1}^n a_k\right)^2 \leq n \sum_{k=1}^n a_k^2$ twice, we have $P_j^2 \leq N^3 \sum_{k=1}^N a_{jk}^4 \varphi_k^{4F}(\xi)$, therefore

$$\int_{\Omega_F} P_j^4 d\xi \leq N^3 \sum_{k=1}^N a_{jk}^4 \int_{\Omega_F} \varphi_k^{4F}(\xi) d\xi \leq N^3 M_F^2 \sum_{k=1}^N a_{jk}^4 \leq N^3 M_F^2 \left(\sum_{k=1}^N a_{jk}^2\right)^2$$

i.e. $\int_{\Omega_F} P_j^4 d\xi \leq N^3 M_F^2 \|P_j\|_{\Omega_F}^4$. Then from this, (9) and (11)

$$(12) \quad \int_{\Omega_F} V^2(X_0, \xi) d\xi \leq N^4 M_F^2 \sum_j \|P_j\|_{\Omega_F}^4 \leq N^4 M_F^2 \left(\int_{\Omega_F} V(X_0, \xi) d\xi\right)^2$$

Since $V^\alpha \in L^2(\Omega_F)$ for all real $\alpha \geq 0$, using Schwarz inequality twice, we obtain:

$$(13) \quad \left(\int_{\Omega_F} V d\xi \right)^{\frac{3}{2}} \leq \int_{\Omega_F} V^{\frac{1}{2}} d\xi \left(\int_{\Omega_F} V^2 d\xi \right)^{\frac{1}{2}}$$

Now using (12) and (13) we have:

$$\left(\int_{\Omega_F} V d\xi \right)^{\frac{1}{2}} \leq N^2 M_F \int_{\Omega_F} V^{\frac{1}{2}} d\xi$$

and from Theorem 7 $M_F \leq \mathbb{C}$ independent of F . Consequently using (8) and the sufficient condition of Theorem 1, we conclude that $-\Delta + \sqrt{V(X)}$ has discrete spectrum. \square

Corollary 11. *Let $V(X)$ be as in Proposition 10. If $-\Delta + V(X)$ has discrete spectrum in $L^2(R^M)$, then $-\Delta + V(X)^{\frac{1}{2n}}$ for $n \geq 1$ natural number, has also discrete spectrum in $L^2(R^M)$.*

Proof. From the two previous inequalities we obtain

$$\left(\int_{\Omega_F} V^{\frac{1}{2n}} d\xi \right)^{\frac{1}{2}} \leq \mathbb{C}_n \int_{\Omega_F} V^{\frac{1}{4n}} d\xi$$

for some $\mathbb{C}_n > 0$. It implies the above result for all $n \geq 1$. \square

In the next proposition we use the following notation:

$$(14) \quad \sum_{M_1, \dots, M_l} := \sum_{M_1=1}^M \sum_{M_2=1}^M \dots \sum_{M_l=1}^M$$

$$(15) \quad \sum_{M_1, \dots, M_l}' := \sum_{M_1=1}^M \sum_{M_2=1}^M \dots \sum_{M_l=1}^M \quad \text{with } M_1 \neq M_2 \neq \dots \neq M_l.$$

Given an index $M_l = k \in 1, 2, \dots, M$ then

$$(16) \quad \sum_{M_1, \dots, M_{l-1}}'' := \sum_{M_1=1}^M \sum_{M_2=1}^M \dots \sum_{M_{l-1}=1}^M \quad \text{with } M_1 \neq M_2 \neq \dots \neq M_{l-1} \neq k$$

Given a set of real coefficients f_{a_1, \dots, a_l}^B where $a_1, \dots, a_l = 1, \dots, N$ we denote

$$(17) \quad \mathcal{F}_{a_1, \dots, a_{l-1}; \widehat{a}_1, \dots, \widehat{a}_{l-1}} = f_{ca_1, \dots, a_{l-1}}^B f_{c\widehat{a}_1, \dots, a_{l-1}}^B + \dots + f_{a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_{l-1}}^B f_{\widehat{a}_1, \dots, \widehat{a}_{i-1}, c, \widehat{a}_{i+1}, \dots, \widehat{a}_{l-1}}^B + \dots + f_{a_1 \dots a_{l-1} c}^B f_{\widehat{a}_1, \dots, \widehat{a}_{l-1}, c}^B,$$

and in general

$$(18) \quad \mathcal{F}_{a_1, \dots, a_{l-i}; \widehat{a}_1, \dots, \widehat{a}_{l-i}} := \sum f_{(c_1, \dots, c_i; a_1, \dots, a_{l-i})}^B f_{(c_1, \dots, c_i; \widehat{a}_1, \dots, \widehat{a}_{l-i})}^B$$

where $(c_1, \dots, c_i; a_1, \dots, a_{l-i})$ denotes a set of l indices. i indices are c_1, \dots, c_i in that order and $l-i$ indices are a_1, \dots, a_{l-i} in that order, the summation is in all possible sets. B is an independent set of indices with any range over which a sum is performed.

We also introduce the matrix \mathcal{M} with components

$$(19) \quad M_{a\hat{a}} := \mathcal{F}_{a;\hat{a}} = f_{c_1, \dots, c_{l-1}, a}^B f_{c_1, \dots, c_{l-1}, \hat{a}}^B + \dots + f_{c_1, \dots, c_{i-1}, a, c_{i+1}, \dots, c_{l-1}}^B f_{c_1, \dots, c_{i-1}, \hat{a}, c_{i+1}, \dots, c_{l-1}}^B + \dots$$

In the proof of the next proposition we will use the following remark.

Remark 12. If $H = -\Delta + X^c X^{\hat{c}} L_{c\hat{c}}$, where $[L_{c\hat{c}}]$ is symmetric and positive then $H \geq \sqrt{\text{tr}[L_{c\hat{c}}]}$. In fact $[L_{c\hat{c}}] = S^T D S$ may be diagonalized and $-\Delta$ is invariant under a rotation $X \rightarrow Y = SY$. We then have

$$(20) \quad H = \sum_m -\frac{\partial^2}{\partial y_m^2} + \lambda_m y_m^2 \geq \sum_i \sqrt{\lambda_i} \geq \sqrt{\text{tr}[L_{c\hat{c}}]}$$

Proposition 13. Let $H = -\Delta + V(X)$ be a Schrödinger operator with potential $V(X)$ given by

$$(21) \quad V(X) = \sum_{M_1, \dots, M_l} \sum_{B=1}^N (X_{M_1}^{a_1} \dots X_{M_l}^{a_l} f_{a_1 \dots a_l}^B)^2$$

let \mathcal{M} be the symmetric matrix defined in (19), $[X_{M_i}^{a_i}] \in \mathbb{R}^{M \times N}$ and f_{a_1, \dots, a_l}^B real coefficients satisfying the following restriction: \mathcal{M} is strictly positive definite. Then H is essentially self adjoint and has a discrete spectrum in $L^2(\mathbb{R}^{M \times N})$.

Remark 14. There is no assumption concerning the symmetry or antisymmetry of f_{a_1, \dots, a_l}^B on the indices a_1, \dots, a_l . It will be clear from the following proof that instead of one B index we may have any number of them.

Proof. We obtain the following inequalities

$$V(X) \geq \sum_{M_1, \dots, M_l} ' \sum_B (X_{M_1}^{a_1} \dots X_{M_l}^{a_l} f_{a_1 \dots a_l}^B)^2 \geq k_0 \sum_{M_l=1}^M X_{M_l}^c X_{M_l}^{\hat{c}} G_{c\hat{c}}^{M_l}$$

for some real number $k_0 > 0$, where

$$G_{c\hat{c}}^{M_l} = \sum_{M_1, \dots, M_{l-1}} '' X_{M_1}^{a_1} \dots X_{M_{l-1}}^{a_{l-1}} \mathcal{F}_{a_1, \dots, a_{l-1}; \hat{a}_1, \dots, \hat{a}_{l-1}} X_{M_1}^{\hat{a}_1} \dots X_{M_{l-1}}^{\hat{a}_{l-1}}$$

does not depend on $X_{M_l}^c$.

For each M_l we have a quadratic potential, we may then use (21) to obtain

$$(22) \quad -\Delta + V(X) \geq \lambda_o \left(-\Delta + \mathbb{C}_o \sqrt{V_1(X)} \right)$$

for some real $\lambda_o > 0$ and $\mathbb{C}_o > 0$. In the same way

$$(23) \quad -\Delta + V_1(X) \geq \lambda_1 \left(-\Delta + \mathbb{C}_1 \sqrt{V_2(X)} \right)$$

where

$$(24) \quad V_1(X) = \sum_{M_1, \dots, M_{l-1}} X_{M_1}^{a_1} \dots X_{M_{l-1}}^{a_{l-1}} \mathcal{F}_{a_1, \dots, a_{l-1}; \hat{a}_1, \dots, \hat{a}_{l-1}} X_{M_1}^{\hat{a}_1} \dots X_{M_{l-1}}^{\hat{a}_{l-1}}$$

$$(25) \quad V_2(X) = \sum_{M_1, \dots, M_{l-2}} X_{M_1}^{a_1} \dots X_{M_{l-2}}^{a_{l-2}} \mathcal{F}_{a_1, \dots, a_{l-2}; \hat{a}_1, \dots, \hat{a}_{l-2}} X_{M_1}^{\hat{a}_1} \dots X_{M_{l-2}}^{\hat{a}_{l-2}}$$

and in general

$$(26) \quad -\Delta + V_i(X) \geq \lambda_i \left(-\Delta + \mathbb{C}_i \sqrt{V_{i+1}} \right)$$

$$V_i(X) = \sum_{M_1, \dots, M_{l-i}} X_{M_1}^{a_1} \dots X_{M_{l-i}}^{a_{l-i}} \mathcal{F}_{a_1, \dots, a_{l-i}; \hat{a}_1, \dots, \hat{a}_{l-i}} X_{M_1}^{\hat{a}_1} \dots X_{M_{l-i}}^{\hat{a}_{l-i}}$$

for some real numbers $\lambda_i > 0$, $\mathbb{C}_i > 0$, $i = 0, \dots, l-2$.

For $i = l-2$ we get

$$(27) \quad -\Delta + V_{l-2}(X) \geq \lambda_{l-2} \left(-\Delta + \mathbb{C}_{l-2} \sqrt{V_{l-1}} \right)$$

and

$$V_{l-1}(X) = \sum_{M_1} X_{M_1}^{a_1} \mathcal{F}_{a_1; \hat{a}_1} X_{M_1}^{\hat{a}_1} = \sum_{M_1} X_{M_1}^a M_{a\hat{a}} X_{M_1}^{\hat{a}}$$

□

This is the potential of an harmonic oscillator in $\mathbb{R}^{M \times N}$, since under the assumption of the Proposition 13 \mathcal{M} is strictly positive. Consequently

$$-\Delta + V_{l-1}(X)$$

has a discrete spectrum in $L^2(\mathbb{R}^{M \times N})$. Using now Proposition (10) we obtain that $-\Delta + \mathbb{C}_{l-2} \sqrt{V_{l-1}}$ has discrete spectrum in $L^2(\mathbb{R}^{M \times N})$ and from (27) $-\Delta + V_{l-2}(X)$ has also discrete spectrum in $L^2(\mathbb{R}^{M \times N})$.

Using this argument several times we conclude that

$$-\Delta + V(X)$$

has a discrete spectrum in $L^2(\mathbb{R}^{M \times N})$.

The property of being essentially self adjoint arises from general arguments for symmetric operators. Moreover H is a positive operator densely defined in $L^2(\mathbb{R}^{M \times N})$, then there exists a positive self adjoint extension of H . It is called the Friedrichs extension of H .

5. CONNECTION WITH ABJM-LIKE THEORIES

In the previous section we rigorously showed at non-perturbative level, a sufficient condition for the discreteness of the spectrum of a Schrödinger operator with a scalar polynomial potential of any degree that can be expressed as the sum of squares. This result generalizes all previously ones in the literature. The requirements for the discreteness are very general and not restricted only to cases of Lie groups or Fillipov algebras expressible as a direct product of Lie algebras, as discussed below. The F 's are any kind of constant that satisfy the regularity condition stated in proposition (13).

This result, which holds for a class of scalar potentials, is far from obvious. There is a fairly widespread belief that positive definite polynomial scalar potentials, which can be expressed as a sum of squared terms, automatically have discrete spectrum. But this is not true. We presented in the introduction an infinite class of such potentials. Consider for instance, the well-known situation for the $D = 11$ supermembrane. The classical theory have unstable solutions with minimal energy. The bosonic spectrum is discrete, in spite of the fact that there are flat directions in the potential (the spectrum is discrete because the Molchanov mean value goes to infinity when one goes to infinity on the configuration space). The supersymmetric

spectrum is continuous from $(0, \infty)$ because the contributions from the spinor terms give rise to an unbounded from below potential, which although is balanced by the $-\Delta$ contribution (the hamiltonian is positive) it allows the wave function to escape to infinity.

It is interesting that there are well-defined sectors of the supermembrane theory (with topological central charge different from zero) which have a discrete spectrum from $(0, \infty)$ with isolated eigenvalues with finite multiplicity [25][26],[27]. We will show in this section that the scalar bosonic potentials for the M-branes, BLG, ABJM and ABJ theories, all have an associated Schrödinger operator with discrete spectrum from zero to infinity, with isolated eigenvalues which have finite multiplicity. We will not consider in this paper the contribution to the potential arising from the Chern-Simons (CS) terms. The qualitative properties of the spectrum of BLG, ABJ/M supersymmetric models are so far unknown. There are three important properties in the general analysis of these last three theories. First the behaviour of the scalar bosonic potential (which is studied in this paper), second the CS contribution, and third the presence of supersymmetry. The supersymmetric interacting terms containing spinor fields in these theories are quadratic on the fields in distinction with the case of the $D = 11$ supermembrane in the Light Cone Gauge (L.C.G) where the dependence is linear. This point may have important consequences in a future analysis of the complete spectrum.

• The BLG case.

To characterize the non-perturbative spectral properties of the scalar potential of BLG/ABJM type, it is necessary first, to formulate these theories in the regularized matrix formalism. These theories have real scalar fields X^{aI} valued in the bifundamental representation of the $\mathcal{G} \times \mathcal{G}'$ algebra, gauge fields A_μ^{ab} where $\mu = 0, 1, 2$ spanning the target-space dimensions, and $a \in \mathcal{G}, b \in \mathcal{G}'$, and spinors $\Psi_{a\alpha}$ also valued in the algebra. Let us consider the sixth degree scalar potential of the BLG case,

$$V = \int dx^3 \frac{1}{12} \text{Tr}([X^I, X^J, X^K])^2 = \int dx^3 \frac{1}{12} f^{abcd} f_d^{efg} (X_a^I X_b^J X_c^K X_e^I X_f^J X_g^K)$$

where f_d^{efg} are the 'structure constants' of the algebra color generators T_a . For the BLG case a 3-algebra relation is satisfied

$$(28) \quad [T^a, T^b, T^c] = f_d^{abc} T^d.$$

We expand now each of the fields X_a^I in a basis of generators T_A ¹, to obtain the regularized model,

$$(29) \quad X_a^I = \sum X_a^{IA} T_A$$

with $A = (a_1, a_2)$. For the enveloping algebra of $su(N)$,

$$(30) \quad T_A T_B = h_{AB}^C T_C, \quad \eta_{AB} = \frac{1}{N^4} \text{Tr}(T_A T_B)$$

h_{AB}^C are given in [33],[23]. We substitute the scalar potential by a regularized one

$$\begin{aligned} V &= \frac{1}{12} f^{abcd} f_d^{efg} X_a^{IA} X_b^{BJ} X_c^{CK} X_e^{EI} X_f^{FJ} X_g^{GK} \text{Tr}_{I_N \times I_N} (T_A T_B T_C T_E T_F T_G) \\ &= \frac{1}{12} f^{abcd} f_d^{efg} h_{AB}^U h_{CE}^V h_{FG}^W h_{UVW} X_a^{IA} X_b^{BJ} X_c^{CK} X_e^{EI} X_f^{FJ} X_g^{GK} \end{aligned}$$

The potential can be re-written as a squared-term

$$(31) \quad V = \frac{1}{12} (F_{\mathcal{U}}^{ABC} X^{\mathcal{A}I} X^{\mathcal{B}J} X^{\mathcal{C}K})^2$$

with coefficients $F_{\mathcal{U}}^{ABC} = f_u^{abc} h_{AB}^E h_{CE}^U$ that do not exhibit antisymmetry in the indices $\mathcal{A} = (A, a), \mathcal{B} = (B, b), \mathcal{C} = (C, c)$ nor are structure constants. Using the Proposition 13 we can assure that this regularized potential has a purely discrete spectrum since

$$(32) \quad f^{abcd} X_a^{IA} = 0 \rightarrow X_a^{IA} = 0. \square$$

The D=11 Supermembrane, the 5-brane, p-branes and the N=8 Bagger Lambert model satisfy the regularity condition for the matrix \mathcal{M} of Proposition 13.

• The ABJ/M case

ABJM theory can be obtained from the 3-algebra expression by relaxing some antisymmetric properties of the 3-algebra structure constant as it is indicated in [7] considering now instead of real scalar fields, -as happens in the BLG case-, complex ones $Z^{a\alpha}$.

In the ABJM case [1], the scalar potential may be re-expressed in a covariant way as a sum of squares [34]. Using the results of [7] where the potential is

$$(33) \quad V = \frac{2}{3} \Upsilon_{Bd}^{CD} \Upsilon_{CD}^{Bd},$$

where

$$(34) \quad \Upsilon_{Bd}^{CD} = f^{ab\bar{c}} Z_a^C Z_b^D \bar{Z}_{B\bar{c}} - \frac{1}{2} \delta_B^C f^{ab\bar{c}} Z_a^E Z_b^D \bar{Z}_{E\bar{c}} + \frac{1}{2} \delta_B^D f^{ab\bar{c}} Z_a^E Z_b^C \bar{Z}_{E\bar{c}}.$$

The zero-energy solutions correspond to $\Upsilon_{Bd}^{CD} = 0$. In distinction with the case of BLG, the ABJM potential includes a sum of three squared terms. The indices

¹In the case of the $D = 11$ supermembrane a regularization procedure in terms of $su(N)$ generators is natural because the structure constants of the $su(N)$ algebra converge in the large N to the structure constants of the area preserving diffeomorphisms, the gauge symmetry of the $D = 11$ Supermembrane in the light cone gauge. In the present case we do not have such argument however following other truncation procedures one can obtain the same result. A straightforward truncation procedure is to expand the fields on a orthonormal basis on the compact base manifold and truncate it at some level in order to have a model which can be analyzed by the rigorous methods of quantum mechanics. As we said the qualitative results are independent of these two regularization procedures.

C, D are mandatory different but not necessarily the index B . We can bound the potential for the one with

$$(35) \quad \Upsilon_{B'd}^{CD} = f^{ab\bar{c}}{}_d Z_a^C Z_b^D \bar{Z}_{B'\bar{c}}$$

where B' is an index different from C, D .

To reduce the analysis to one in quantum mechanics, a regularization procedure is performed. The regularity condition of Proposition 13, in terms of the triple product [7], may be expressed as a

$$(36) \quad [X, T^b; \bar{T}^{\bar{c}}] = f^{ab\bar{c}}{}_d X_a T^d = 0 \quad \forall b, \bar{c} \Rightarrow X_a = 0.$$

Note that if this condition is not satisfied, the potentials we are considering have continuous spectrum. This result follows using Molchanov, Maz'ya and Schubert theorem. Factorizing out the constants due to regularization process, in the case of ABJM and ABJ it follows from (49) in [7] that (36) implies²

$$(37) \quad (t_\alpha^\lambda)^{a\bar{c}} X_a = 0,$$

where t_α^λ are $u(N)$ representations of the gauge algebra \mathcal{G} . In the case \mathcal{G} is $u(N)$ then the regularity condition is satisfied.

The proposition (13) in our paper ensures then that the Schrödinger operator associated to the regularized scalar sixth degree potential of ABJM has also purely discrete spectrum. \square

This proves a necessary condition for quantum stability for the new supersymmetric models. In fact, a continuous spectrum at the regularized bosonic model arising from a formulation on a compact space, would imply several difficulties on the models. For example, the Feynmann kernel would be ill defined. This is the first step in order to consider a non perturbative analysis of these new supersymmetric models.

• Some more comments.

If we now add the regularized Chern-Simons gauge contribution V_{CS} to the scalar potential there are quadratic and cubic contributions. Without loss of generality, take for simplicity the BLG case. Although the shape of CS terms clearly do not suit in the shape of the potentials here considered, one could imagine to bound this potential $V_{sixth} + V_{CS}$, however one can realize that the cubic contribution is not necessarily positive. A gauge fixing procedure must be performed here in order to analyze the problem. At this stage we cannot guarantee the discreteness of the regularized bosonic potential once the gauge fields are added and a further study is needed in this approximation. However the main point concerning the stability of the bosonic multiple branes is the analysis of the V_{sixth} .

Another interesting issue is the spectral characterization of the complete hamiltonians including their supersymmetric extension. The analysis then, is much more involved. All of these actions of multiple M2's have in common the construction of a conformal supersymmetric gauge theory with quadratic couplings in the fermionic variables. It goes like combinations of terms of the type $\bar{\Psi}^\dagger (\Gamma X X^\dagger) \Psi$. Their

²The condition holds in spite that the polynomial expression considered in equation (49) of [7] for the $f^{ab\bar{c}d}$ in terms of $u(N)$ generators is not the final one, since antisymmetry in the two first indices still needs to be imposed by hand as the authors explain. This fact does not affect the present condition (37), since antisymmetric solutions represent a subset of solutions we consider here.

fermionic contribution in the light cone Hamiltonian formulation [35], in distinction with the case of a single M2 brane, still depends quadratically on the bosonic variables. The sufficient condition for discreteness of supersymmetric potentials shown in [25] is no longer applicable and although it does not exclude completely the possibility of the spectrum be discrete at regularized non-perturbative level, makes it much more fine tuned.

6. CONCLUSIONS

We obtain a general class of polynomials for which the Schrödinger operator has a discrete spectrum. This class includes all the scalar potentials in membrane, 5-brane, p-branes, multiple M2 branes, BLG and ABJM theories. We provide a proof of the discreteness of the spectrum of the associated Schrödinger operators. This is a first step in order to analyse BLG and ABJM super-symmetric theories from a non-perturbative point of view. This proves a necessary condition for quantum stability for the new supersymmetric models. In fact, a continuous spectrum at the regularized bosonic model would imply several difficulties on the models. For example, the Feynmann kernel would be ill defined. This is the first step in order to consider a non perturbative analysis of these new supersymmetric models.

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